$$\mathcal{X}(\Omega, C) = \{x : \Omega \to C\}$$

$$(\alpha x + \beta y)(u) = \alpha x(u) + \beta y(u)$$

for all $u \in \Omega$

$$\langle x, y \rangle = \int_\Omega \langle x(u), y(u) \rangle_C \, d\mu(u)$$

$$g(\Omega) \to x'$$

Symmetry

Transformation that leaves a certain property of said object or system unchanged or invariant.

Group

Set G with a binary operation

$$\circ : G \times G \to G$$

$g \circ h$

Represented henceforth as gh

Axioms

1) $(gh)l = g(hl) \quad \forall g, h, l \in G$ → Associativity

2) Identity $e \in \Omega$ → Existence of identity

$eg = ge = g \quad \forall g \in G$

3) Inverse — Existence of inverse

$gg^{-1} = e$

4) Closure → Closure

$g \in \Omega,\ h \in \Omega \Rightarrow gh \in G$

Commutative groups — Abelian

Group action

Group action of G on a set Ω

$x \in \mathcal{X}(\Omega) \qquad u \in \Omega$

$(g \cdot x)(u) = x(g^{-1}u)$ →

Linear group actions — Group representations

Geometric deep learning

- $f : \mathcal{X}(\Omega) \longrightarrow \mathcal{Y}$ is G-invariant

$$f(g \cdot x) = f(x) \text{ for all } g \in G$$

Example

Translation group

$$G = \{(-1,-1), (-1,0), (0,-1), (0,0), (1,1), (0,1), (1,0)\}$$

$g \cdot x$ - 2D shift of an image

$$f(g \cdot x) = f(x)$$

$\Longrightarrow$ Detect the "cat" even if it is shifted.

Similarly, G - equivariant $\mathcal{X} \to \Omega \to \mathcal{C}$

$$f : \mathcal{X}(\Omega) \longrightarrow \mathcal{X}(\Omega)$$

$$f(g \cdot x) = g \cdot f(x) \quad \forall g \in G$$

Group - Convolution (Definition)

$$(x * \theta)(g) = \int_{\Omega} x(u)\, \theta(g^{-1}u)\, du$$

G - equivariant

$g \in G,\ h \in G$

Proof

Refer Geometric deep learning - Bronstein

Too difficult as the group can be very big

e.g.

1) all rotations, translations, reflections

2) all permutations of the graphs

Spectral interpretation of Group Convolution

$$y_0 = c_0 x_0 + c_1 x_1 + c_2 x_2$$

Facts

1) Any circular convolution in 1D represented as a circular convolution

circulant matrix

$$\boxed{y = \begin{bmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \quad \begin{matrix} x_2 \\ x_0 \\ x_1 \end{matrix}}$$

2) Eigenvectors of ALL CIRCULANT MATRICES

- Fourier basis

$$\Phi = (\varphi_0, \ldots, \varphi_{n-1})$$

$$\varphi_k = \frac{1}{\sqrt{n}} \left(1, e^{i\frac{2\pi k}{n}}, e^{i\frac{4\pi k}{n}}, \ldots, e^{i\frac{2\pi(n-1)k}{n}}\right)$$

1D convolution in spatial domain

$$x * \theta = \Phi \begin{bmatrix} \hat{\theta}_0 & & & \\ & \hat{\theta}_1 & & \\ & & \ddots & \\ & & & \hat{\theta}_{n-1} \end{bmatrix} \Phi^* x$$

Finally, graphs - Nodes N (1 to n)

$X - [\underbrace{x_0}_{\mathbb{R}^d} \; x_1 \; \dots \; x_n] - \mathbb{R}^{d \times n}$

$A - \mathbb{R}^{n \times n}$ adjacency matrix

$D^{-1/2} \overset{A}{L} D^{-1/2}$

D - degree matrix (Diagonal)

$L = (D - A)$ Laplacian matrix

$f(X) = \underset{d \times n}{X} \underset{n \times n}{L}$ (permutation equivariant)

f will be permutation equivariant

Key idea: Choose eigenvectors V of L as spatial basis

L

$L' = P^T L P$

X

$X' = XP$

$PP^T = I$

$X' L' = (XL) P$

Spectral graph convolution

$$f_\theta(X) = U \begin{bmatrix} \hat{\theta}_0 & & \cdots \\ & \hat{\theta}_1 & \\ & & \ddots \\ \cdots & & & \hat{\theta}_2 \end{bmatrix} U^T X$$

θ - learnable parameters

Can have additional non-linearities

* This was initially called a graph convolution but is uncommon now

Reasons

1) Cannot generalize to different graph structures
2) Very difficult to scale

MORE COMMON (Spatial Graph convolution)

Somewhat similar to CNN over local neighborhoods

- What Jonathan and Ting-Han presented

Bridging the Gap Between Spectral and Spatial Domains in Graph Neural Networks

Muhammet Balcilar*, Guillaume Renton, Pierre Héroux, Benoit Gaüzère, Sébastien Adam, Paul Honeine

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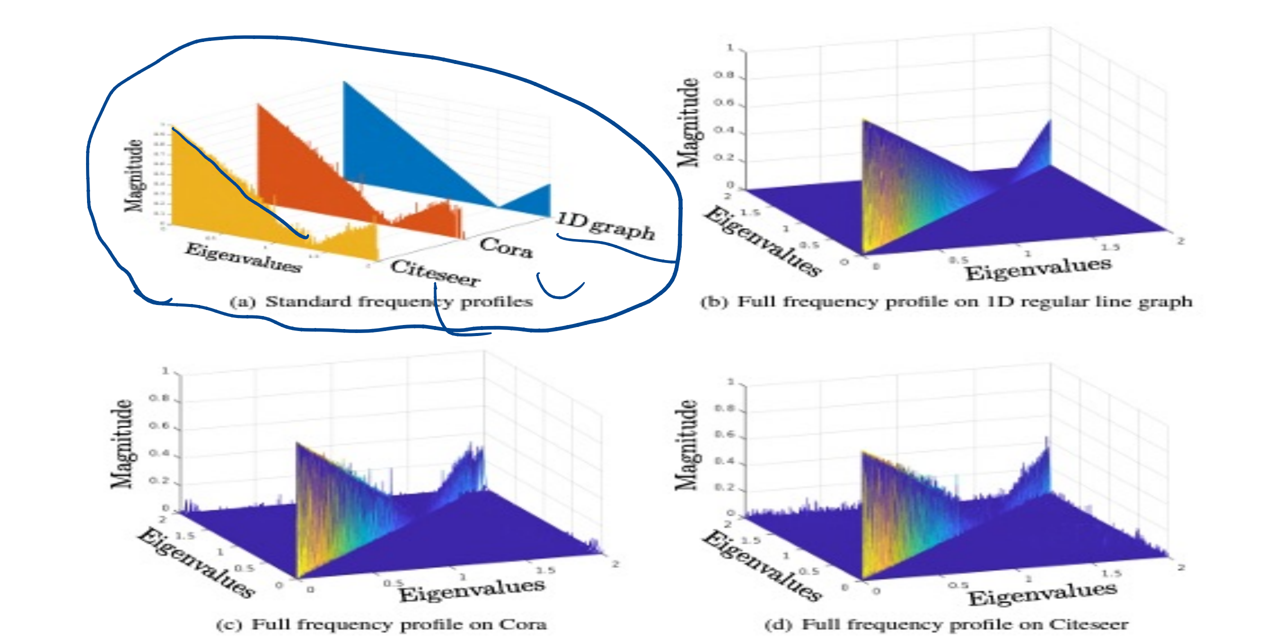

Figure 4: Frequency profiles of GCN on different graphs.

$$f(X) = \sigma\left(X\left(I + A\right)\right)$$

$$= \sigma\left(\sum_s \underbrace{C_s}_{(I+A)} X \underbrace{W}_{\text{learnable weights}}\right)$$

Preceding neural networks

CNN – translation equivariance

Spherical CNN – equivariance on SO(3)
rotation

GNN – permutation equivariance

Transformers – special case of GNN

Geodesics and gauges – ?